Weinberg
2.2

$F^{\mu\nu} \to \Lambda^T F^{\mu\nu} \Lambda$

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$$

The wave vector is in $\vec{y}$, polarization (by convention is $\vec{E}$) is in $\vec{z}$, so $\vec{B}$ is in direction $\vec{k} \times \vec{E} = \vec{y} \times \vec{z} = \vec{x}$.
Then we have $F^{\mu\nu}$

$$F^{\mu\nu}_{(\text{as observed by } O)} = \begin{pmatrix} 0 & 0 & 0 & -E_z \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -B_0 \\ E_z & 0 & B_0 & 0 \end{pmatrix}$$

$$\Lambda = \text{boost in } \vec{z} = \begin{pmatrix} \gamma & 0 & 0 & -\gamma_\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma_\beta & 0 & 0 & \gamma \end{pmatrix}$$

$F^{\mu\nu} = \partial_\mu A^\nu - \partial_\nu A^\mu$, where $A^\mu = (\Phi/c, \vec{A})$
transforms as a tensor, we wish to find $F'^{\mu\nu}$ as observed by O', this will give us a complete description of the EM wave in O' frame.

$$F'^{\mu\nu} = \Lambda^T F^{\mu\nu} \Lambda$$

$$= \begin{pmatrix} \gamma & & & -\gamma\beta \\ & 1 & & \\ & & 1 & \\ -\gamma\beta & & & \gamma \end{pmatrix} \begin{pmatrix} 0 & & & -E_0 \\ & 0 & & \\ & & 0 & -B_0 \\ E_0 & & B_0 & 0 \end{pmatrix} \begin{pmatrix} \gamma & & & -\gamma\beta \\ & 1 & & \\ & & 1 & \\ -\gamma\beta & & & \gamma \end{pmatrix}$$

$$= \begin{pmatrix} \gamma & & & -\gamma\beta \\ & 1 & & \\ & & 1 & \\ -\gamma\beta & & & \gamma \end{pmatrix} \begin{pmatrix} \gamma\beta E_0 & 0 & 0 & -\gamma E_0 \\ 0 & 0 & 0 & \\ \gamma\beta B_0 & 0 & 0 & -\gamma B_0 \\ \gamma E_0 & 0 & B_0 & -\gamma\beta E_0 \end{pmatrix}$$

$$= \begin{pmatrix} \gamma^2\beta E_0 - \gamma^2\beta E_0 & 0 & -\gamma\beta B_0 & -\gamma^2 E_0 + \gamma^2\beta^2 E_0 \\ 0 & 0 & & 0 \\ \gamma\beta B_0 & 0 & & -\gamma B_0 \\ -\gamma^2\beta^2 E_0 + \gamma^2 E_0 & 0 & \gamma B_0 & \gamma^2\beta E_0 - \gamma^2\beta E_0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & -\gamma\beta B_0 & \gamma^2(\cancel{1-\beta^2}\ \beta^2-1) E_0 \\ 0 & 0 & 0 & 0 \\ \gamma\beta B_0 & 0 & 0 & -\gamma B_0 \\ \gamma^2(1-\beta^2)E_0 & 0 & \gamma B_0 & 0 \end{pmatrix}$$

The corresponding $\vec{E}', \vec{B}'$ are

$$\vec{E}' = (0 , \gamma\beta B_0 , \gamma^2(1-\beta^2) E_0)$$

$$\vec{B}' = (\gamma B_0 , 0 , 0)$$

$\gamma = \sqrt{\frac{1}{1-\beta^2}} \Rightarrow \gamma^2(1-\beta^2) = 1$, then

$$\vec{E}' = (0, \gamma\beta B_0, E_0)$$

$$\vec{B}' = (\gamma B_0, 0, 0)$$

Comparing this with the original polarization $\vec{E} = (0, 0, E_0)$, we see that the polarization direction has changed, it now gained an $\vec{y}$ component, now in y-z plane. Since $\vec{B}'$ is still in the $\vec{x}$ plane, $\vec{E}'$ and $\vec{B}'$ remain orthogonal, we are good, but this implies the wave vector $\vec{k}'$ must differ from $\vec{k} \propto \hat{y}$.

As a check, we compute $\vec{k}' = \vec{E}' \times \vec{B}'$

$$= -\gamma^2 \beta B_0^2 \hat{z} + \gamma B_0 E_0 \hat{y}$$

$$\Rightarrow \vec{k}' = (0, \gamma B_0 E_0, -\gamma^2 \beta B_0^2).$$

$B_0 = \frac{1}{c^2} E_0$, in our formalism, $c = 1 \Rightarrow B_0 = E_0$

This renders the direction of $\vec{k}'$ as

$$\hat{k}' \propto (0, \gamma E_0^2, -\gamma^2 \beta E_0^2)$$

$$\Rightarrow \boxed{\hat{k}' \propto (0, 1, -\gamma\beta).}$$

This would be consistent if $(\frac{\omega}{c}, \vec{k})$ is a 4-vector:

$$k'^{\mu} = \Lambda k^{\mu} = \begin{pmatrix} \gamma & & & -\gamma\beta \\ & 1 & & \\ & & 1 & \\ -\gamma\beta & & & \gamma \end{pmatrix} \begin{pmatrix} \frac{\omega}{c} \\ \vec{k} \end{pmatrix}$$

$$= \begin{pmatrix} |k|\gamma \\ 0 \\ |k| \\ -|k|\gamma\beta \end{pmatrix}$$

which also gives $\boxed{\hat{k}' \propto (0, 1, -\gamma\beta)}$

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